the electrostatic selfenergy. Therefore:

$$\int_{-\infty}^{+\infty} f \,\mathrm{d}u \cdot \int_{-\infty}^{+\infty} f \,\mathrm{d}u \rightarrow \tfrac{1}{2} \left( \int_{-\infty}^{+\infty} f \,\mathrm{d}u \right)^2 \quad (4.14)$$

but since $\left( \int_{-\infty}^{+\infty} f(u,t) \,\mathrm{d}u \right)^2 = k^4 \varphi^2 / (4 \pi e)^2$ (4.15)

we have $\tfrac{1}{2} m \langle u^2 \rangle - k^2 \varphi^2 / 8 \pi n = \tfrac{1}{2} m \langle u^2 \rangle_{t=0}$ (4.16)

or $\tfrac{1}{2} m \langle u^2 \rangle + E^2 / 8 \pi n = \tfrac{1}{2} m \langle u^2 \rangle_{t=0}$. (4.17)

From the generalized temperature defined by Eq. (2.30) we have:

$$\varkappa T = \varkappa T_0 + m \langle u^2 \rangle - m \langle u \rangle^2, \quad (4.18)$$

for $t = 0$ we have in particular:

$$\varkappa T_0 = [\varkappa T]_{t=0} \quad (4.19)$$

therefore: $\langle u^2 \rangle_{t=0} = \langle u \rangle^2_{t=0} = u_0^2$. (4.20)

From (4.17) follows therefore:

$$m \langle u^2 \rangle = m u_0^2 - E^2 / 4 \pi n . \quad (4.21)$$

Or making use of (3.45):

$$m \langle u^2 \rangle = m u_0^2 (1 - e^{-2\gamma t} \sin^2 \omega_p t) . \quad (4.22)$$

On the other hand from (4.9) it follows that:

$$m \langle u \rangle^2 = m u_0^2 \, e^{-2\gamma t} \cos^2 \omega_p t . \quad (4.23)$$

Inserting (4.22) and (4.23) into (4.18) we finally have:

$$\varkappa T = \varkappa T_0 + m u_0^2 (1 - e^{-2\gamma t}) \quad (4.24)$$

which becomes

$$T / T_0 = e^{2(S - S_0)/n\varkappa} = 1 + (m u_0^2 / \varkappa T_0) \ (1 - e^{-2\gamma t}) . \quad (4.25)$$

At $t = \infty$ all the energy is converted into thermal energy:

$$\varkappa T_\infty = \varkappa T_0 + m u_0^2 . \quad (4.26)$$

In Fig. 5 a plot is given for the change in the generalized temperature as a function of time.

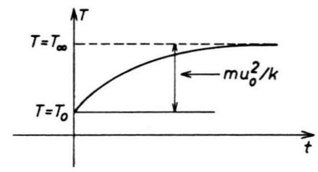


Fig. 5. Time-dependence of the generalized temperature for LANDAU damping of electrostatic plasma waves.

From Eq. (4.25) we conclude that for LANDAU damping the generalized temperature $T$ is a steadily increasing function of time. The same applies to $S$, which is a satisfactory behavior for a quantity called entropy.

We would like to add the remark that for an initial disturbance in the distribution function which has the form of a $\delta$-function $f(u,0) = A\,\delta(u - u_0)$ one obtains the undamped VAN KAMPEN modes. Only in this special case $T = \text{const}$ and therefore also $S = \text{const}$.

# Effects of Magnetic Shear on Density Gradient Drift Instabilities *

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The amounts of magnetic shear necessary to stabilize density gradient drift instabilities are estimated for a plasma with $1 \gg \beta > m/M$ where $\beta$=plasma pressure/magnetic pressure, $m$ and $M$ are the electron and ion masses. The stability criteria for electrostatic convective modes are shown to be slightly modified from those obtained previously for $\beta < m/M$. For $\beta > m/M$ however, we must also consider unstable ALFVÉN type modes. It is shown that the critical amount of magnetic shear is proportional to $\beta^{-1}$ for the convective ALFVÉN type modes and to $\sqrt{\beta}$ for the non-convective modes.

## § 1. Introduction

Because of the necessity of stabilization of high temperature plasmas in thermonuclear devices, the effects of magnetic shear on drift instabilities due to spatial inhomogeneity have recently been discussed extensively for both collisional and collisionless plasmas [1–7]. In almost all of these discussions, espe-

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cially for collisionless plasmas, only the electrostatic perturbations have been considered assuming a low $\beta$ situation $\beta<m/M$, where $\beta$ is the ratio of plasma to magnetic pressure, and $m$ and $M$ are the electron and ion masses. For $\beta>m/M$, however, such an electrostatic approximation is not precise and we must also consider the perturbation of the magnetic field, especially when the phase velocity along the unperturbed magnetic field is of the order of the ALFVÉN speed. In the present paper, therefore, the amounts of magnetic shear necessary to stabilize drift instabilities due only to density gradient are evaluated for a plasma in which $\beta$ satisfies

$$m/M<\beta\ll 1\,.$$

Since we assume $\beta\ll 1$, we can neglect the compression of the magnetic field and we consider only the bending as a perturbation of the magnetic field. In the absence of magnetic shear, it is known that there exist two unstable modes in such a plasma: one of them corresponds to the ion sound wave and the other to the slow ALFVÉN wave (see Appendices A and B).

As has been shown in the previous works, the sheared magnetic field has two main effects which tend to stabilize drift instabilities. One of them is an effective increase in the ion LANDAU damping available for stabilization: (a) through the propagation of the wave packet [6] or (b) through the expansion of the "potential" well, into the region where the LANDAU damping by the interaction with resonant ions is strong. The other is the shrinking of the localized region of the wave packet until it can not exist [1, 2].

In the next section, the methods to be used in the present paper are discussed by briefly reviewing those used in the previous works (see also Appendix C). In § 3, it is shown that the stability criteria for convective electrostatic modes are only slightly modified from the previous ones obtained for $\beta<m/M$. In § 4, stability criteria are derived for both convective and non-convective unstable ALFVÉN modes. It is found there that the critical amount of shear for stabilization decreases with $\beta$ for the convective ALFVÉN mode and increases with $\sqrt{\beta}$ for the non-convective one.

In Appendix A, we derive the dispersion relation in the case without magnetic shear using a simple method different from the previous ones. The dispersion relation obtained in Appendix A is used in §§ 2–4 with some modifications introduced by the magnetic shear. In Appendix B, the analysis of the drift instability is briefly given. In refs. [1, 2] and [5], some stability criteria for non-convective electrostatic modes have been derived. However, since their derivations, in particular, by Russian authors might be difficult for the readers, we derive them in Appendix C in which we use the dispersion relation obtained in Appendix A.

## § 2. Method

We consider a plasma with a density gradient $dn/dx$ in the $x$ direction and a magnetic field $B_0\,\boldsymbol{e}_z$ along the $z$ axis. Using the VLASOV and MAXWELL equations, the dispersion equation for low frequency modes in the absence of magnetic shear is given by [8, 9] (see also Appendix A):

$$\left\{(\omega+\omega_e)\left(1-i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\right)-\frac{k_z^2 v_i^2}{2\omega^2}(\omega-\omega_e)\,I_0(b)\,e^{-b}\right\}\times\left\{\omega(\omega-\omega_e)-\frac{b}{1-I_0(b)\,e^{-b}}k_z^2 v_A^2\right\}=b\,k_z^2 v_A^2(\omega-\omega_e)\,,\tag{2.1}$$

where $v_e$ and $v_i$ are the electron and ion thermal velocities, $b=a^2(k_x^2+k_y^2)/2$, $a$ is the ion gyro-radius, $v_A$ is the ALFVÉN speed, $I_0$ is the modified BESSEL function of the first kind, and

$$\omega_e=\left|\frac{k_y\,c\,T}{e\,B_0}\frac{1}{n}\frac{dn}{dx}\right|.$$

The following seven assumptions have been used:

(1) the time-space dependence of perturbations varies as

$$\exp(i\omega t+i\boldsymbol{k}\boldsymbol{r})\,,$$

(2) $k\lambda_D\ll 1$ where $\lambda_D$ is the DEBYE length,

(3) $k_x\gg r^{-1}=\left|\dfrac{dn}{dx}\Big/n\right|$,

(4) the electron gyro-radius is much shorter

[8] A. B. MIKHAILOVSKII and L. I. RUDAKOV, Zh. Eksperim. Teor. Fiz. 44, 912 [1963] (Soviet Phys.—JETP 17, 621 [1963]).

[9] A. B. MIKHAILOVSKII, Voprosy Teorii Plazmy (Problems of Plasma Theory) Vol. 3, Atomizdat [1963].

than the wavelength,

(5) $k_\perp = \sqrt{k_x^2 + k_y^2} \gg k_z$,

(6) $v_i \ll |\omega/k_z| \ll v_e$, and

(7) $\omega \ll \Omega_i =$ the ion gyro-frequency.

Also the local approximation has been used, in which the particle density and its gradient $dn/dx$ are treated as constant. As is shown in Appendix B, Eq. (2.1) gives us two unstable modes: one of them corresponds to the ion sound wave and the other to the slow ALFVÉN wave.

In considering the influence of magnetic shear, as usually done, we take an unperturbed magnetic field of the form $\boldsymbol{B} = B_0 \boldsymbol{e}_z + (x/L_s) B_0 \boldsymbol{e}_y$ where $L_s$ is the shearing distance. Then the shear effects can be taken into account by the substitution of $k_\parallel = k_z + (x/L_s) k_y$ instead of $k_z$ in the dispersion equation (2.1)[2,6]. Solving Eq. (2.1) about $b \neq 0$[10], we can estimate $k_x(x, \omega)$ and also the amplitude of wave packets after their propagation over a certain distance[6] (for convective modes). On the other hand, the substitution of $\partial/\partial x$ for $i k_x$ gives us a differential equation for the perturbations[2]. The differential equation in the small or large wavelength limit usually has a form similar to the SCHRÖDINGER equation with a complex "potential". Now our problem is to find the finite solution for unstable perturbations (for non-convective modes).

As mentioned below, the treatments of the previous studies on the shear effects may be classified into two or three kinds according to the character of the unstable modes. Russian authors[1,2] have derived some stability criteria fro localized unstable modes. They used the fact that the potential well shrinks by the introduction of magnetic shear and the localized unstable solution cannot exist if the width of the localized region becomes narrower than the wavelength along the $x$ axis. MIKHAILOVSKAYA and MIKHAILOVSKII[1] and also GALEEV[2] obtained the following stability criterion for electrostatic modes:

$$L_s/r < (r/a)^{2/3}. \tag{2.2}$$

They considered a temperature gradient as well as a density gradient. From other considerations, they both obtained another criterion which is essentially the same in both papers:

$$L_s/r < (r/a)^{1/2} \beta^{-1/4}. \tag{2.3}$$

We can see that Eqs. (2.2) and (2.3) are independent of the temperature gradient. In Appendix C, we also derive them for our case without a temperature gradient.

KRALL and ROSENBLUTH[5] derived a different stability criterion for $\beta < m/M$

$$L_s/r < 2\sqrt{2}(r/a). \tag{2.4}$$

In deriving this criterion, they considered a condition that the oscillating region of the "potential" expands, by the introduction of shear field, into the region in which the ion LANDAU damping prevails. The above three criteria have been derived by considering a spatial variation of $\omega_e$. It seems curious that Eqs. (2.2) and (2.3) are derived considering the contraction of the potential well by the magnetic shear, while Eq. (2.4) is derived considering the expansion. In Appendix C, we derive the above three criteria and discuss them.

The two methods mentioned above are concerned with non-convective unstable modes and are applicable in cases in which two or more turning points are found. We must still consider convective unstable modes if they exist. COPPI et al.[6] recently investigated the shear effects for convective-type unstable electrostatic modes in a plasma with a density gradient and $\beta < m/M$. Considering a convective solution with a form

$$\varphi(x,t) = \varphi_0 \int f(\omega) \exp\{i \int^x k_x(\omega, x)\, dx + i\omega t\}\, d\omega$$

and choosing $f(\omega) = \exp\{-(\omega-\omega_0)^2 T^2/2\}$ with $\omega_0 T \gg 1$, they obtained a general stability criterion

$$-\int^x \mathrm{Im}\, k_x\, dx < n_0. \tag{2.5}$$

Here $\exp(n_0)$ represents a tolerable growth of the convective mode during its propagation towards the region in which the ion LANDAU damping predominates, and the limits of integration should be determined from the assumptions used in solving for $k_x(x, \omega)$. It is necessary for the application of Eq. (2.5) that no STOKES phenomena occur along the path of integral.

In the following two sections, we shall derive stability criteria for convective modes by use of Eq. (2.5). In § 4, we also derive a stability criterion for non-convective ALFVÉN modes by considering the contraction of the potential well.

[10] For $b = 0$, we have no unstable drift waves provided $\nabla T = 0$.

## § 3. Electrostatic Modes

If we assume $|\omega(\omega-\omega_e)| \ll k_{||}^2 v_A^2$ in Eq. (2.1), we have the dispersion equation for electrostatic drift waves. As is shown in Appendix A, this approximation is valid for $\beta < m/M$ which is equivalent to $2 v_A^2 > v_e^2$. In the long wavelength limit ($b \ll 1$), where we have $|\omega| \sim \omega_e$, remembering the assumption $v_i \ll |\omega/k_{||}| \ll v_e$, we can be sure that the neglect of the perturbed magnetic field is always valid for $\beta < m/M$. For $\beta > m/M$, however, the electrostatic approximation is allowed only under the condition

$$v_i \ll |\omega/k_{||}| \ll v_A . \qquad (3.1)$$

In the long wavelength case, from Eq. (2.1) we obtain approximately

$$b = -\frac{\omega+\omega_e}{\omega-\omega_e}\left(1 - i\sqrt{\pi}\frac{\omega}{|k_{||}|v_e}\right) + \frac{k_{||}^2 v_i^2}{2\omega^2}, \qquad (3.2)$$

where we put $k_{||} = (x/L_s)\, k_y$ for $k_z$ in order to consider the shear effect. This is naturally the same as Eq. (16) of reference [6] except for the condition Eq. (3.1). Explicitly taking into account Eq. (3.1) we briefly repeat the calculation similar to that done in ref. [6]. Putting $i k_x = \partial/\partial x$ in Eq. (3.2), we obtain a differential equation for the perturbations. Under the present limitation of constant density gradient, there exist no non-convective normal modes, for which the shear stabilization was considered by KRALL and ROSENBLUTH [5]. Then we consider the stability of convective modes by use of Eq. (2.5). Neglecting $k_y$ compared with $k_x$ on the left-hand side of Eq. (3.2), we obtain the amplification during its propagation

$$-\int^x \mathrm{Im}\, k_x \,\mathrm{d}x \qquad (3.3)$$
$$= \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{M}{m}}\frac{L_s}{r}\,\varpi\,\sqrt{\left|\frac{1-\varpi}{1+\varpi}\right|}\,\ln\left(\frac{\xi_{max}}{\xi_{min}}\right),$$

where

$$\varpi = |\omega/\omega_e| \quad \text{and} \quad \xi = (2\, r\, x / a\, L_s\, \varpi).$$

The limits of the integration are determined by Eq. (3.1) and also by the condition of the neglect of the ion LANDAU damping.

$$\xi^2_{max} = \frac{1}{\ln\left\{\sqrt{\frac{M}{m}}\left|\frac{1+\varpi}{1-\varpi}\right|\right\}} > \xi^2 > \beta = \xi^2_{min}. \qquad (3.4)$$

We must notice that no STOKES phenomena [11] occur along the integral path.

Substituting $\varpi \sim (\sqrt{5}-1)/2$ corresponding to the maximum amplification, we obtain a stability criterion

$$\frac{L_s}{r} < 12\sqrt{\frac{M}{m}}\, n_0\, A, \qquad (3.5)$$

where

$$A^{-1} = \ln(\xi^2_{max}/\xi^2_{min}) = \ln\left\{\frac{1}{\beta\ln(4\sqrt{M/m})}\right\}.$$

As expected, the stability criterion is only a little different from the previous one for $\beta < m/M$.

In the short wavelength limit ($b \ll 1$), only the electrostatic modes with $|\omega| \ll \omega_e$ are found to be possible and their dispersion equation is given by Eq. (A.32) in Appendix B. In the derivation of the stability criterion, the integration limits in Eq. (2.5) are hardly affected by an additionl condition

$$|\omega(\omega-\omega_e)| \ll k_{||}^2 v_A^2.$$

Therefore, the stability criterion is not altered from the previous one obtained for [6] $\beta < m/M$ and we need not write it down.

## § 4. Alfvén Modes

As is shown in Appendix B, the unstable ALFVÉN modes exist in a inhomogeneous plasma with a density gradient provided $v_A \lesssim |\omega_{k||}|$ [12]. Till now, only a few authors have discussed the shear stabilization of these modes. For a plasma with a temperature gradient as well as a density gradient, GALEEV [2] derived a criterion of shear stabilization for such modes. He took into account the spatial variation of $(\mathrm{d}n/\mathrm{d}x)/n$. The unstable mode which he considered is only a non-convective mode. In the present paper, $\omega_e$ is assumed to be constant. As we see below, both convective and non-convective unstable ALFVÉN modes are possible. First we consider the shear stabilization of convective modes.

### *(A) Convective Modes*

In the long wavelength case, the dispersion equation Eq. (2.1) is reduced to

$$\left\{(\omega+\omega_e)\left(1 - i\sqrt{\pi}\frac{|k_{||}|v_e}{\omega}\right) - \frac{k_{||}^2 v_i^2}{2\omega^2}(\omega-\omega_e)\right\}$$
$$\{\omega(\omega-\omega_e) - k_{||}^2 v_A^2\} = b\, k_{||}^2 v_A^2 (\omega-\omega_e), \qquad (4.1)$$

where we use $k_{||} = k_z + (x/L_s)\, k_y$ instead of the previous $k_z$ in order to consider the shear effect.

[11] E. C. KEMBLE, The Fundamental Principles of Quantum Mechanics, McGraw-Hill, New York 1937.

[12] We must notice that the unstable ALFVÉN type modes exist even in plasmas with a density gradient only.

We consider wave packets localized in a small region $\delta$ around $x=0$ such that $k_y \delta/L_s$ is negligible compared with constant $k_z$. Then, two turning points which may allow the existence of non-convective modes cannot be found in a region $0<x<x_m$ where $x_m$ is defined by

$$(x_m/L_s)\, k_y = k_z\,.$$

When the wave packet travels a distance $x_m$, we might regard the instability as being destroyed. The reason is that after it travels $x_m$, Eq. (4.1) is satisfied only by $b \sim 1$ which is inconsistent with the assumption $b \ll 1$ used in deriving Eq. (4.1). Even in the case that this picture fails, however, there is a possibility that the following consideration appears to be reasonable. That is, the time $t_T$ in which the wave packets traverses $x_m$ is longer than an operating period of plasma devices. In this case the wave packet will be reflected at most once from the turning point. The group velocity in the $x$ direction can be estimated to be roughly $\omega_e/k_x$. Thus we have

$$t_T \sim \frac{L_s}{v_d}\,\frac{k_z k_x}{k_y^2} \quad \text{where} \quad v_d = \omega_e/k_y\,.$$

For example, considering Eq. (4.5) derived below, we obtain

$$t_T \lesssim \frac{k_z k_x}{k_y^2}\,\frac{r^2}{v_i\, a}\left(\frac{M}{\pi m}\right)^{1/2} 10\, n_0\,.$$

If we assume $r=10$ cm, $v_i = 10^7$ cm/sec, $a=1$ cm and

$$10\, n_0 \left(\frac{k_z k_x}{k_y^2}\right) \cong 1-10\,,$$

we have $$t_T \lesssim 10^{-4}-10^{-3}\ \text{sec}\,.$$

Following the method used in reference [6] we discuss the stabilization of such convective Alfvén type modes.

The frequency of the unstable Alfvén mode is roughly given by

$$\omega_0 = \frac{\omega_e}{2} - \sqrt{\left(\frac{\omega_e}{2}\right)^2 + k_z^2 v_A^2}.$$

Then, putting $\omega = \omega_0 + \delta\omega$ in Eq. (4.1), we obtain

$$-\operatorname{Im} k_x = -\frac{\sqrt{\pi}}{a}\,\frac{\omega_0}{k_z v_e}\,\frac{1}{k_z v_A}\sqrt{\frac{\omega_0+\omega_e+\delta\omega}{|\omega_0-\omega_e|}}\left\{\delta\omega\sqrt{\left(\frac{\omega_e}{2}\right)^2 + k_z^2 v_A^2} + \frac{k_y}{k_z}\,\frac{x}{L_s}\,k_z^2 v_A^2\right\}^{1/2}, \tag{4.2}$$

where we neglected $k_y$ compared with $k_x$ in $b = \frac{1}{2} a^2 (k_x^2 + k_y^2)$.

Regarding $\delta\omega$ as a parameter, we can avoid the difficulty arising from the strong $x$-dependence of the frequency. Integrating Eq. (4.2) with respect to $x$ gives us the amplification of the wave packet during its propagation

$$-\int_0^{x_m} \operatorname{Im} k_x\, dx = -\frac{\sqrt{\pi}}{a}\,\frac{\omega_0}{k_z v_e}\,\frac{1}{k_z v_A}\sqrt{\frac{\omega_0+\omega_e+\delta\omega}{|\omega_0-\omega_e|}}\int_0^{x_m}\left\{\delta\omega\sqrt{\left(\frac{\omega_e}{2}\right)^2 + k_z^2 v_A} + + \frac{k_y}{k_z}\,\frac{x}{L_s}\,k_z^2 v_A^2\right\}^{1/2} dx\,. \tag{4.3}$$

We use a general criterion of shear stabilization of convective modes, Eq. (2.5), given in reference [6].

For simplicity, we first consider the case in which $2\,\omega_e^2 \approx k_z^2 v_A^2$. In this case, the frequency $\omega_0$ of the unstable Alfvén mode is equal to that of the electrostatic mode and is given by $\omega_0 = -\omega_e$. After the integration over $x$, taking a value of $\delta\omega$ corresponding to the maximum amplification, we obtain a stability criterion

$$L_s/r < 12\, n_0 (M/\pi m)^{1/2}. \tag{4.4}$$

As $\operatorname{Im} k_x$ is nearly proportional to $\omega_0$, the amount of shear necessary for stabilization is expected to become smaller for smaller values of $\omega_0$. In fact, for

$$\omega_e^2 > 4\, k_z^2 v_A^2 \quad \text{where} \quad \omega_0 \approx -\omega_e (k_z v_A/\omega_e)^2,$$

similar calculations give us

$$-\int \operatorname{Im} k_x\, dx = \frac{\sqrt{\pi}}{6}\,\frac{L_s}{r}\sqrt{\frac{m}{M}}\left(\frac{k_z v_A^2}{\omega_e}\right)^2 = \frac{\sqrt{\pi}}{6}\,\frac{L_s}{r}\sqrt{\frac{m}{M}}\left(\frac{k_z}{k_y}\right)^2\left(\frac{2r}{a}\right)^2\frac{1}{\beta}. \tag{4.5}$$

Thus we have a stability criterion

$$\frac{L_s}{r} < 6\, n_0 \sqrt{\frac{\pi m}{M}}\left(\frac{\omega_e}{k_z v_A}\right)^2 = \frac{3}{4}\, n_0 \left(\frac{r}{a}\right)^2\left(\frac{k_y}{k_z}\right)^2\sqrt{\frac{M}{\pi m}}\,\beta. \tag{4.6}$$

Then we find that the amount of magnetic shear for the stabilization of convective ALFVÉN type modes becomes smaller for a plasma with higher $\beta$. The above treatment may also be applied to the case where $b$ is not much smaller than unity, but the calculation will become complicated.

### *(B) Non-convective Localized Modes*

Regarding $\omega$ as determined by an eigenvalue equation, we can also consider non-convective localized modes. Neglecting the small imaginary part and replacing $k_x$ by $-i(\partial/\partial x)$ in Eq. (4.1), we have a differential equation like SUYDAM's [13]:

$$\left[\frac{d^2}{dx^2} - \left\{k_y^2 - \frac{2}{a^2}\frac{\omega+\omega_e}{\omega-\omega_e}\left[\frac{\omega(\omega-\omega_e)}{k_{\parallel}^2 v_A^2} - 1\right]\right\}\right]\varphi(x) = 0, \tag{4.7}$$

where $k_{\parallel} = (x/L_s)\, k_y$. This equation has a singularity at $x=0$. However, it has been derived under the assumption $|\omega/k_{\parallel} v_e| \ll 1$ and is not applicable for small $x$ such that $|\omega/k_{\parallel} v_e| \gg 1$. For such a region around $x=0$, which we denote region I, using the asymptotic formula of the $W$ function for both electron and ion, we have a differential equation [see Eq. (A.20)]

$$\left[\frac{d^2}{dx^2} - \left\{k_y^2 + \frac{1}{a^2}\frac{M\beta}{2m}\left(1+\frac{\omega_e}{\omega}\right) - \frac{1}{a^2}\frac{M\beta}{2m}\frac{\omega+\omega_e}{\omega-\omega_e}\frac{k_y^2 v_A^2}{\omega^2}\frac{L_s^2}{x^2}\right\}\right]\varphi(x) = 0. \tag{4.8}$$

For an intermediate region of $x$ (region II), in which we do not have any good approximation formula of the electron $W$ function, we must interpolate from both sides to obtain an adequate equation. However, its derivation is not necessary for our dicussion. The "potential" $U(x)$ of Eq. (4.7) which is valid in region III has two turning points

$$\pm x_1 = \pm \sqrt{\omega(\omega-\omega_e)/k_y^2 v_A^2}\, L_s .$$

Since we consider the non-convective mode, we must have a potential well between the two turning points. Since we have assumed $b \ll 1$ in the derivation of Eq. (4.8), the inequality

$$1 \gg \frac{\omega+\omega_e}{\omega-\omega_e} > 0$$

must also be fulfilled. Then we have $\omega < -\omega_e$ and $|\omega| \approx \omega_e$. Thus we find

$$x_1 \approx L_s (v_i/v_A)\,(a/r).$$

Because $\omega < -\omega_e$, there exists a potential hill in region I, whose width is so narrow that the last term in Eq. (4.8) can be neglected. The neglect of the last term is consistent with the assumption $|\omega/k_{\parallel} v_e| > 1$. Then Eq. (4.8) is reduced to

$$\left[\frac{d^2}{dx^2} - \left\{k_y^2 + \frac{M\beta}{2ma^2}\left(1+\frac{\omega_e}{\omega}\right)\right\}\right]\varphi(x) = 0. \tag{4.8'}$$

We can draw a schematic profile of the potential $U(x)$ as shown in Fig. 1. The inner turning points $\pm x_2$ may be taken as $|\omega/k_{\parallel} ve| \approx 1$. Thus we have

$$x_2 \approx L_s (v_i/v_e)\,(a/r).$$

When the shear increases, the width of the potential well shrinks and becomes narrower than the wavelength $\lambda_x$. Even if the potential hill in region I is transparent, the localized solution can not exist provided $2x_1 < \lambda_x$. Then we have a sufficient condition for stabilization

$$\frac{L_s}{r}\frac{v_i}{v_A}\frac{a}{r} < \lambda_x$$

which is rewritten

$$L_s/r < C/\sqrt{\beta}, \tag{4.9}$$

where $$C^2 = \frac{\omega-\omega_e}{\omega+\omega_e} \gg 1 .$$

Whether the potential hill is transparent is not important for the above derivation. However, if

$$L_s/r > C/\sqrt{\beta},$$

we can easily find that the hill is not transparent and that its width is too large to use the connection formulas. The connection formulas at $x = \pm x_2$ can be used only if the relation

$$\int_{-x_2}^{x_2} \sqrt{U(x)}\, dx > 1$$

is satisfied, from which we obtain

$$L_s/r > C/\beta\sqrt{m/M}.$$

Therefore, if $L_s/r > C/\sqrt{\beta}$, finite solutions exist. That is, the criterion Eq. (4.9) is the necessary and sufficient condition for shear stabilization. This is also derived by use of the SUYDAM type of analysis.

[13] B. R. SUYDAM, Proc. 2nd Conf. U.N. on Peaceful Uses of Atomic Energy, Geneva **31**, 137 [1958].

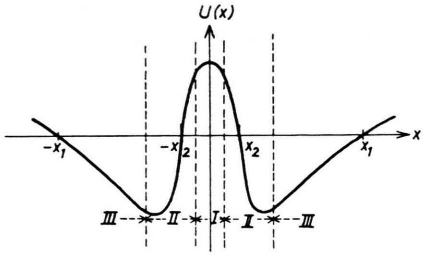


Fig. 1. Profile of the potential $U(x)$.

The formal solution of Eq. (4.7) is given by

$$\varphi(x) = \text{const} \cdot \sqrt{x}\, K_{i\nu}(\alpha x), \qquad (4.10)$$

where $K_{i\nu}$ is the modified BESSEL function of the second kind,

$$\nu^2 = \chi^2 - 1/4,$$

$$\chi^2 = \frac{2 L_s^2}{a^2} \frac{\omega(\omega + \omega_e)}{k_y^2 v_A^2},$$

and

$$\alpha^2 = k_y^2 + \frac{2}{a^2} \frac{\omega + \omega_e}{\omega - \omega_e}.$$

If $\nu^2 < 0$, there exists no finite solution. From $\nu^2 < 0$, we obtain the condition of shear stabilization Eq. (4.7). GALEEV[2] obtained a similar criterion for a plasma with a temperature gradient as well as a density gradient. However, his criterion is not applicable in our case.

The stability criterion contains a factor $C$ larger than unity. The evaluation of $C$ is difficult in our present treatment. It should be estimated by a more quantitative treatment, for example, by numerically solving an integral equation which will be obtained by taking into account more rigorously the $x$ dependence of the perturbations.

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## Appendix A

### *Derivation of the Dispersion Relation*

In this appendix, using a rather simple method compared with the previous ones, we rederive a known dispersion relation for low frequency waves, including electromagnetic modes, in a spatially inhomogeneous plasma with a density gradient $dn/dx$ in the $x$ direction, a magnetic field $\boldsymbol{B} = B_0(x)\, \boldsymbol{e}_z$ along the $z$ direction and no magnetic shear. The equilibrium distribution function, satisfying the VLASOV equation and MAXWELL's equation, is taken as

$$f_{0j} = n(\alpha_j/\pi)^{3/2} \exp(-\alpha_j v^2) \left\{1 - \frac{1}{h_j}\left(x + \frac{v_y}{\Omega_j}\right)\right\}, \qquad (A.1)$$

where $j$ denotes the species of particles;

$$\alpha_j = m_j/2\,T_j = 1/v_j^2$$

in which $v_j$, $m_j$ and $T_j$ are the thermal velocity, mass and temperature of the $j$ species;

$$\Omega_j = e_j B_0/m_j c, \quad \text{and} \quad h_j = -\frac{1}{n_j}\frac{dn_j}{dx}.$$

From the MAXWELL equations

$$\nabla \times \boldsymbol{B}_0 = \frac{4\pi}{c} \sum_j e_j \int f_{0j}\, \boldsymbol{v}\, d^3v,$$

$$\nabla \times \boldsymbol{E}_0 = 4\pi \sum_j e_j \int f_{0j}\, d^3v = 0$$

we find $h_e = h_i \equiv r$ and

$$\frac{dB_0}{dx} = \frac{4\pi n}{r B_0}(T_e + T_i).$$

In order to get the dispersion relation for the case where $m_e/m_i < \beta \ll 1$, it is rather convenient to introduce the vector potential $A\,\boldsymbol{e}_z$ parallel to the unperturbed magnetic field as well as the scalar potential $\varphi$ for the perturbed fields $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$.

$$\boldsymbol{E}_1 = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}\boldsymbol{e}_z,$$

$$\boldsymbol{B}_1 = \nabla \times (A\,\boldsymbol{e}_z).$$

The above choice of the vector potential means that for $m_e/m_i < \beta \ll 1$ we consider only bendings of the magnetic field of the perturbations and that we can neglect the compression. We can also neglect a drift due to $dB_0/dx$ in a case of such low $\beta$.

$\varphi$ and $A$ satisfy the equations

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -4\pi\varrho, \qquad (A.2)$$

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A = -\frac{4\pi}{c} j_z. \qquad (A.3)$$

Here $\varrho$ is the charge density and $j_z$ the current density along the $z$ direction

$$\varrho = \sum_j e_j \int f_{1j}\, d^3v, \qquad (A.4)$$

$$j_z = \sum_j e_j \int f_{1j}\, v_z\, d^3v, \qquad (A.5)$$

where $f_{1j}$ is a small perturbation from the equilibrium distribution and is given by solving the linearized VLASOV equation

$$\frac{\partial f_{1j}}{\partial t} + \boldsymbol{v}\cdot\nabla f_{1j} + \frac{e_j}{m_j c}\,\boldsymbol{v}\times\boldsymbol{B}_0\cdot\nabla_{\boldsymbol{v}} f_{1j} = -\frac{e_j}{m_j}\left(\boldsymbol{E}_1 + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}_1\right)\cdot\nabla_{\boldsymbol{v}} f_{0j}\,. \tag{A.6}$$

We solve Eq. (A.6) by the method of characteristics

$$f_{1j} = -\frac{e_j}{m_j}\int\left(\boldsymbol{E}_1 + \frac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}_1\right)\cdot\nabla_{\boldsymbol{v}} f_{0j}\,, \tag{A.7}$$

where the integration is carried out along the particle orbit in the lowest order. We assume a time-space dependence of the perturbations as follows:

$$\begin{Bmatrix} f_{1j} \\ \varphi \\ A \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{1j} \\ \bar{\varphi} \\ \bar{A} \end{Bmatrix} \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} + \omega t)\}$$

where $\bar{\varphi}$, $\bar{A}$ and $\bar{f}_{1j}$ are treated as constant in the lowest order. We are interested in a weekly inhomogeneous plasma and the present considerations are restricted to $k_x r \gg 1$.

Substituting the perturbed distribution function Eq. (A.7) into Eqs. (A.2) and (A.3), we obtain the homogeneous equations for $\bar{\varphi}$ and $\bar{A}$

$$\sum_j k_{\mathrm{d}j}^2 \{1 - (\omega - \omega_j)\,\Phi_{1j}\}\,\bar{\varphi} + \frac{1}{c}\sum_j k_{\mathrm{d}j}^2\,(\omega - \omega_j)\,\Phi_{2j}\,\bar{A} = 0\,, \tag{A.8}$$

$$-\frac{1}{c}\sum_j k_{\mathrm{d}j}^2\,(\omega - \omega_j)\,\Phi_{2j}\,\bar{\varphi} + \left\{k^2 + \frac{1}{c^2}\sum_j k_{\mathrm{d}j}^2\,(\omega - \omega_j)\,\Phi_{3j}\right\}\bar{A} = 0\,, \tag{A.9}$$

where

$$\begin{aligned} k_{\mathrm{d}j}^2 &= 4\pi n_j e_j^2/T_j\,, \qquad \omega_j = k_y/2\,\alpha_j\,\Omega_j\,r\,,\\ k_\perp^2 &= k_x^2 + k_y^2, \qquad b = k_\perp^2\,v_j^2/2\,\Omega_j^2,\\ \Phi_{ij} &= \sum_{n=-\infty}^{\infty} \frac{I_n(b_j)\,e^{-b_j}}{\omega + n\,\Omega_j}\left\{1 + W\left(\frac{\omega + n\,\Omega_j}{k_z v_j}\right)\right\}, \end{aligned} \tag{A.10}$$

$$\Phi_{2j} = -\frac{1}{k_z}\sum_{n=-\infty}^{\infty} I_n(b_j)\,e^{-b_j}\,W\left(\frac{\omega + n\,\Omega_j}{k_z v_j}\right), \tag{A.11}$$

$$\Phi_{3j} = \frac{1}{k_z^2}\sum_{n=-\infty}^{\infty} (\omega + n\,\Omega_j)\,I_n(b_j)\,e^{-b_j}\,W\left(\frac{\omega + n\,\Omega_j}{k_z v_j}\right), \tag{A.12}$$

$$W(z) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{t\,e^{-t^2}}{t+z}\,\mathrm{d}t = \begin{cases} i\sqrt{\pi}\,z\exp(-z^2) + \dfrac{1}{2z^2} + \dfrac{3}{4z^4} + \ldots, & |z| \gg 1 \\ i\sqrt{\pi}\,z\exp(-z^2) - 1 + 2z^2\,(1 - \frac{2}{3}z^2 + \ldots), & |z| \ll 1\,, \end{cases} \tag{A.13}$$

and we assumed $\omega^2 \ll k^2 c^2$, $k^2 \ll k_{\mathrm{d}j}^2$, and $\varrho = 0$ which mean the quasi-neutrality of charge. We see that Eq. (A.8) without the second term is the usual dispersion relation of the longitudinal electrostatic mode.

Since we consider low frequency perturbations $|\omega| \ll \Omega_i$, all terms with $n \neq 0$ can be neglected. Then we have the dispersion relation

$$\begin{aligned} &\left[\sum_j k_{\mathrm{d}j}^2\left\{1 - \left(1 - \frac{\omega_j}{\omega}\,(1 + W_j)\,I_0(b_j)\,e^{-b_j}\right\}\right]\left[k^2 + \frac{\omega^2}{c^2 k_z^2}\sum_j\left(1 - \frac{\omega_j}{\omega}\right)W_j\,I_0(b_j)\,e^{-b_j}\right] \\ &\qquad + \frac{\omega^2}{c^2 k_z^2}\,k_{\mathrm{d}j}^2\left[\sum k_{\mathrm{d}j}^2\left(1 - \frac{\omega_j}{\omega}\right)W_j\,I_0(b_j)\,e^{-b_j}\right]^2 = 0\,, \end{aligned} \tag{A.14}$$

which is easily reduced to the next form

$$A\left(\frac{\omega^2}{c^2 k_z^2}\,B - k^2\right) + k^2 B = 0\,. \tag{A.15}$$

Here

$$A = \sum_j k_{\mathrm{d}j}^2\left(1 - \frac{\omega_j}{\omega}\right)W_j\,I_0(b_j)\,e^{-b_j}\,, \tag{A.16}$$

and

$$B = \sum_j k_{\mathrm{d}j}^2\left\{1 - \left(1 - \frac{\omega_j}{\omega}\right)I_0(b_j)\,e^{-b_j}\right\}. \tag{A.17}$$

We must remark that Eq. (A.15) is also applicable for plasmas consisting of many species.

If we further assume $v_i \ll |\omega/k_z| \ll v_e$ and $b_e \ll 1$, Eq. (A.15) is reduced to

$$\left\{\left(1-\frac{\omega_e}{\omega}\right)\left(1-i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\right)-\frac{T_e}{T_i}\left(1-\frac{\omega_i}{\omega}\right)\frac{k_z^2 v_i^2}{2\omega^2}I_0(b)\,e^{-b}\right\} \times\left\{\omega(\omega-\omega_i)-\frac{k^2}{k_\perp^2}\frac{b\,k_z^2 v_A^2}{1-I_0(b)\,e^{-b}}\right\}=\frac{k^2}{k_\perp^2}b\,k_z^2 v_A^2\frac{T_e}{T_i}\left(1-\frac{\omega_i}{\omega}\right), \tag{A.18}$$

where we use a relation $k_e^2/c^2 k_\perp^2=(T_i/T_e)/b\,v_A^2$, in which $b\equiv b_i$. When we assume $k_\perp \gg k_z$ and put $k^2/k_\perp^2=1$, Eq. (A.18) is reduced to the dispersion relation obtained by MIKHAILOVSKII [9].

For the same conditions, we can also write the dispersion relation as follows:

$$2-\left(1-\frac{\omega_i}{\omega}\right)I_0\,e^{-b}\left(1+\frac{k_z^2 v_i^2}{2\omega^2}\right)-i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\left(1-\frac{\omega_e}{\omega}\right) -\frac{\omega^2}{k_z^2 v_i^2}\frac{\beta}{2b}\left(1-\frac{\omega_i}{\omega}\right)\left(1-\frac{\omega_i}{\omega}\right)\left(1-i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\right)\left\{1-I_0(b)\,e^{-b}\left(1+\frac{k_z^2 v_i^2}{2\omega^2}\right)\right\}=0, \tag{A.19}$$

where we assume further $T_e=T_i$ and also we use the relations $v_i^2/v_A^2=\beta/2$ and $c^2 k_\perp^2/k_z^2=b\,v_A^2$. This is the expression obtained by KADOMTSEV [14]. If we neglect the last term in Eq. (A.19), it gives the usual dispersion relation of the electrostatic drift wave for $\beta<m/M$ where $m\equiv m_e$ and $M\equiv m_i$. In Eq. (A.19), we can see that when $\beta<m/M$ the last term is always negligible and that when $\beta>m/M$ we can not always neglect the last term.

For $v_i \ll v_e \ll |\omega/k_z|$, using the asymptotic formula of the $W$ functions for both electron and ion, we obtain a dispersion relation

$$\left(1+\frac{\omega_e}{\omega}\right)\left\{\frac{k_z^2 v_e^2}{2\omega^2}+i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\exp\left(-\frac{\omega^2}{k_z^2 v_e^2}\right)\right\}\times\left\{\omega(\omega-\omega_e)-\frac{b\,k_z^2 v_A^2}{1-I_0(b)\,e^{-b}}\right\}=b\,k_z^2 v_A^2\left(1-\frac{\omega_e}{\omega}\right), \tag{A.20}$$

where we assume $T_e=T_i$ and we rewrite $\omega_e=|k_y/2\,\alpha_e\,\Omega_e\,r|$ instead of $\omega_e=k_y/2\,\alpha_e\,\Omega_e\,r$. In the text and also in Appendices B and C, we use this new definition of $\omega_e$.

## Appendix B

### *Analysis of Stability*

On the basis of the dispersion relation obtained in Appendix A, we discuss some limiting solutions obtained for long and short wavelength. Here we assume equal electron and ion temperatures, and we also assume $v_i \ll |\omega/k_z| \ll v_e$.

### *1. Long Wavelength Case* ($b \ll 1$)

In this case, using an approximation

$$I_0(b)\,e^{-b}=1-b,$$

Eq. (A.18) is reduced to

$$\left\{(\omega+\omega_e)\left(1-i\sqrt{\pi}\frac{\omega}{|k_z|v_e}\right)-\frac{k_z^2 v_i^2}{2\omega^2}(\omega-\omega_e)\right\} \times\{\omega(\omega-\omega_e)-k_z^2 v_A^2\}=b\,k_z^2 v_A^2(\omega-\omega e), \tag{A.21}$$

from which we can find three modes: one of them corresponds to the ion sound mode and the other two to the slow and fast ALFVÉN modes.

#### 1. a) Electrostatic Modes

When $\omega^2 \ll k_z^2 v_A^2$ we find a nearly electrostatic mode, the frequency of which is roughly equal to $-\omega_e$. Then, putting $\omega=-\omega_e+\delta\omega$ in Eq. (A.21), we can find easily

$$\delta\omega=\omega e\left\{\frac{2b}{1-(2\,\omega_e^2/k_z^2 v_A^2)}-\left(\frac{k_z v_i}{\omega}\right)^2\right\} \times\left(1-i\sqrt{\pi}\frac{\omega_e}{|k_z|v_e}\right). \tag{A.22}$$

Therefore, the instability condition which means $\mathrm{Im}\,\omega<0$ is written as

$$\frac{2b}{1-(2\,\omega_e^2/k_z^2 v_A^2)}>\left(\frac{k_z v_i}{\omega_e}\right)^2 \tag{A.23}$$

[14] B. B. KADOMTSEV, Zh. Eksperim. Teor. Fiz. **45**, 1230 [1963] (Soviet Phys.—JETP **18**, 847 [1964]).

or $\quad k_z^2 v_A^2 > 2\,\omega_e^2$ and $2\,b \gtrsim \left(\frac{k_z v_i}{\omega}\right)^2$.

The last inequality is rewritten as

$$k_\perp^2 \gtrsim k_z^2 (\Omega_i/\omega_e)^2 \gg k_z^2,$$

which is consistent with the initial assumption, and it means that the unstable electrostatic modes propagates almost perpendicularly to the magnetic field because waves with $k_\perp \sim k_z$ are strongly damped by the ion LANDAU damping.

### 1. b) ALFVÉN Modes

Under the condition that the second bracket in the left-hand side of Eq. (A.21) is roughly equal to zero, we get the slow and fast ALFVÉN waves whose frequencies are roughly given by the solutions of $\omega(\omega-\omega_e)-k_z^2 v_A^2=0$:

$$\omega_2 = \frac{\omega_e}{2} - \sqrt{\left(\frac{\omega_e}{2}\right)^2 + k_z^2 v_A^2}, \qquad \omega_3 = +\frac{\omega_e}{2}\sqrt{\left(\frac{\omega_e}{2}\right)^2 + k_z^2 v_A^2}. \tag{A.24, 25}$$

Substituting $\omega \equiv \omega_{2,3} + \delta\omega$ into Eq. A.18) with $T_e = T_i$ and using the approximate formula $I_0(b)\,e^{-b} = 1 - b + \frac{4}{3} b^2$, we get

$$\delta\omega = -\frac{2\,b\,k_z^2 v_A^2}{3\sqrt{(\omega_e/2)^2 + k_z^2 v_A^2}} - \frac{b\,k_z^2 v_A^2 (\omega_2-\omega_e)(1+i\sqrt{\pi}\,\omega_z/|k_z|\,v_e)}{2(\omega_2+\omega_e)\sqrt{(\omega_e/2)^2+k_z^2 v_A^2}} \tag{A.26}$$

for the slow wave, and

$$\delta\omega = \frac{2\,b\,k_z^2 v_A^2}{3\sqrt{(\omega_e/2)^2 + k_z^2 v_A^2}} + \frac{b\,k_z^2 v_A^2 (\omega_3-\omega_e)(1+i\sqrt{\pi}\,\omega_3/|k_z|\,v_A)}{2(\omega_3+\omega_e)\sqrt{(\omega_e/2)^2+k_z^2 v_A^2}}, \tag{A.27}$$

for the fast wave.

From Eqs. (A.26) and (A.27) we can easily find that the instability condition for the slow wave is

$$2\,\omega_e^2 > k_z^2 v_A^2$$

and that the fast wave is always stable under the present assumptions.

For $k_z v_A \approx \sqrt{2}\,\omega_e$, the frequency of the electrostatic mode is nearly equal to that of the slow ALFVÉN mode, and Eqs. (22) and (26) are not valid. Substituting $k_z v_A = \sqrt{2}\,\omega_e$ in Eq. (A.21), therefore, we obtain

$$\omega = -\omega_e + \delta\omega \tag{A.28}$$

and $\quad (\delta\omega)^2 = \frac{2}{3}\,b\,k_z^2 v_A^2 \left(1 - i\sqrt{\pi}\,\frac{\omega_e}{|k_z|\,v_e}\right)$.

In the case where $|\omega/k_z| \gg v_e$, similarly we can also analyse the stability by use of Eq. (A.20) and find the same unstable modes, but with exponentially small growth rates.

### 2. *Short Wavelength Case* $(b>1)$

We use the asymptotic formula

$$I_0(b)\,e^{-b} \approx 1/\sqrt{2\pi b}\,.$$

When $|\omega| \ll \omega_e$, as we can see a posteriori, we obtain the relevant dispersion relation from Eq. (A.18) or from Eq. (A.19)

$$2 + \frac{\omega_e}{\omega}\left(1 + \frac{k_z^2 v_i^2}{2\,\omega^2}\right)\frac{1}{\sqrt{2\pi b}} + \frac{\omega_e^2}{k_z^2 v_i^2}\frac{\beta}{2\,b} - i\sqrt{\pi}\,\frac{\omega_e}{|k_z|\,v_e} = 0\,. \tag{A.29}$$

In order to obtain this result from Eq. (A.18), we must retain the first term of the second bracket on the left-hand side and we may say that this wave corresponds to the ALFVÉN mode. Solving Eq. (A.29), we obtain

$$\omega = -\frac{\omega_e}{\sqrt{2\pi b}}\,\frac{1}{2 + \frac{\omega_e^2}{k_z^2 v_i^2}\frac{\beta}{2\,b} - i\sqrt{\pi}\,\frac{\omega_e}{|k_z|\,v_e}}, \tag{A.30}$$

which is rewritten as

$$\omega = -\frac{(k_y/k_\perp)\,r^{-1}(T/2\pi M)^{1/2}}{2 + \left(\frac{k_y}{k_\perp}\right)^2 \frac{\beta}{4\,r^2 k_z^2} - i\sqrt{\pi}\,\frac{\omega_e}{|k_z|\,v_e}}\,. \tag{A.31}$$

Therefore we can see that this mode is unstable.

When the first term of the second bracket on the left-hand side of Eq. (A.18) is negligible compared with the second term, we obtain a slightly different frequency

$$\omega = -\frac{\omega_e}{\sqrt{2\pi b}}\,\frac{1}{2 - i\sqrt{\pi}\,\frac{\omega_e}{|k_z|\,v_e}}, \tag{A.32}$$

which corresponds to the unstable electrostatic mode.

## Appendix C

*Shear Stabilization of Non-convective Electrostatic Modes*

Several authors [1, 2, 5] have derived criteria for the shear stabilization of non-convective electrostatic modes by taking into account the spatial variation of $\omega_e$. Here, utilizing the results obtained in Appendix A, the previous criteria are derived and compared with one another.

For simplicity, we assume equal electron and ion temperatures. From Eq. (A.18) we have a differential equation for the perturbations by use of the method mentioned in the text

$$\left\{\frac{d^2}{dx^2} - U(x)\right\}\varphi(x) = 0, \tag{A.33}$$

where

$$U(x) = k_y^2 + \frac{2}{a^2}\left\{\frac{\omega+\omega_e}{\omega-\omega_e}\left(1 - i\sqrt{\pi}\,\frac{\omega}{|k_\parallel|\,v_e}\right) - \frac{k_\parallel^2 v_i^2}{2\,\omega^2}\right\}\left\{1 - \frac{\omega(\omega-\omega_e)}{k_\parallel^2 v_A^2}\right\}. \tag{A.34}$$

Here we assumed $b \ll 1$ and $v_i \ll |\omega/k_z| \ll v_e$. Since we consider only the electrostatic modes, we can take

$$1 > \left|\frac{\omega(\omega-\omega_e)}{k_\parallel^2 v_A^2}\right|.$$

The factor $$1 - \frac{\omega(\omega-\omega_e)}{k_\parallel^2 v_A^2}$$

is considered only for the determination of turning points of the "potential" $U(x)$.

### 1. Krall–Rosenbluth *Stability Criterion*

Krall and Rosenbluth considered a spatial variation of $\omega_e$ and took $x=0$ at a point where $\omega_e$ has a maximum value. Expanding $U(x)$ about $x=0$, we have

$$U(x) = \frac{2}{a^2}\left\{\frac{\omega+\omega_*}{\omega-\omega_*} - \frac{x^2}{2}\left(\frac{k_y^2 v_i^2}{\omega^2 L_s^2} - \frac{\omega_*''}{\omega-\omega_*}\right)\right\} \tag{A.35}$$

where we write $\omega_* = \omega_e(x=0)$ and drop the factor

$$1 - \frac{\omega(\omega-\omega_e)}{k_\parallel^2 v_A^2}.$$

[15] Krall and Rosenbluth considered that the potenial has a hill in a small region around $x=0$ under the condition (A.37).

In order that a potential well exist around $x=0$ in the absence of shear,

$$\frac{\omega+\omega_*}{\omega-\omega_*} < 0, \tag{A.36}$$

from which we find $|\omega| < \omega_*$. The shape of the "potential" changes by the introduction of shear [15].

For higher values of magnetic shear, such as

$$\left(1 - \frac{\omega_*}{\omega}\right)\frac{k_y^2 v_i^2}{\omega^2 L_s^2} > \frac{\omega_*''}{\omega}, \tag{A.37}$$

the coefficent of $x^2$ is negative. Then the width of the potential well increases until $U(x)$ becomes positive again, as we can see using the approximate formula of the ion $W$ function for small argument

$$\left|\frac{\omega}{k_\parallel v_i}\right| < 1.$$

In the region of large $x$ values,

$$|\omega/k_\parallel v_i| < 1,$$

the ion Landau damping is strong enough to suppress the instability. Because $\omega \cong -\omega_e$ we can rewrite (A.37) as follows:

$$(L_s/r)^2 < 8(r/a)^2$$

or

$$L_s/r < 2\sqrt{2}(r/a), \tag{A.38}$$

which is taken as a stability criterion.

If (A.37) or (A.38) is not satisfied, the coefficient of $x^2$ in $U(x)$ is positive. Krall and Rosenbluth mentioned that for small shear,

$$\left(1 - \frac{\omega_*}{\omega}\right)\frac{k_y^2 v_i^2}{\omega^2 L_s^2} < \frac{\omega_*''}{\omega},$$

the instability persists although with a slightly altered growth rate.

### 2. Mikhailovskaya–Mikhailovskii–Galeev *Stability Criteria*

Mikhailovskaya and Mikhailovskii [1], and also Galeev [2] derived independently two stability criteria. One side of the region of the localization is limited by $|\omega/k_\parallel| < v_A$. From this limitation, we have a turning point at

$$x_A \approx a\sqrt{\beta}\,L_s/r.$$

Another turning point which might be taken far from $x_A$ can be determined from the condition that the factor

$$\frac{\omega + \omega_e}{\omega - \omega_e} - \frac{k_\parallel^2 v_A^2}{2\,\omega^2}$$

has an extreme value.

Then we obtain a possible maximum width of the potential well

$$\Delta x \approx \frac{1}{8} \left(\frac{L_s}{r}\right)^2 \frac{a^2}{r}.$$

If $\Delta x < \lambda_x \approx a \sqrt{r/\Delta x}$, that is, $\Delta x < a^{2/3} r^{1/3}$, the unstable localized solution cannot exist. Then we have a stability criterion

$$L_s/r < 2\sqrt{2}\,(r/a)^{2/3}. \qquad (A.39)$$

Mikhailovskaya and Mikhailovskii [1] also obtained another stability criterion

$$L_s/r < (r/a)^{1/2} \beta^{-1/4}. \qquad (A.40)$$

Beyond the turning point $x = x_A$, the localized solution is expressed by use of the Airely function [16]

$$\varphi(x) \sim \left(\frac{a^2}{U'(x_A)}\right)^{1/4} (x_A - x)^{-1/4} \times \exp\left\{-\frac{2}{3}\left(\frac{U'(x_A)}{a^2}\right)^{1/2} (x_A - x)^{3/2}\right\}, \qquad (A.41)$$

where $U'(x_A)$ is roughly given by $\Delta x / r\, x_A$, that is,

$$U'(x_A) \sim \frac{1}{8} \frac{L_s\, a}{r^3 \sqrt{\beta}}.$$

Then the characteristic length of exponential decay which we denote by $\delta x$ is given by

$$\delta x \sim 2\,(a\, r^3 \sqrt{\beta}/L_s)^{1/3}.$$

Finally, we can derive the stability criterion Eq. (A.40) from the condition $\delta x > x_A - x_e$ where $x_e$ is defined by $\omega_e \approx k_\parallel v_e$.

Galeev [2] also obtained almost the same criterion as Eq. (A.40). He mentioned that if the transparency condition of a hill between $x_A$ and $x_e$ is satisfied, that is, if

$$\int_{x_e}^{x_A} \sqrt{U(x)}\, dx < 1, \qquad (A.42)$$

the region of existence of the localized solution expands into the ion Landau damping region. The transparency condition Eq. (A.42) is essentially the same as $\delta x > x_A - x_e$. However, his statement is not right because the ion Landau damping region does not lie in a region

$$x < x_e (< x_A),$$

but in a region

$$x \sim x_i \gg x_A$$

where $x_i$ is defined $|k_\parallel v_i| = \omega_e$. Eq. (A.40) means that the localized region of the perturbation extends into the region of the electron Landau damping.

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# A General Formulation for the Evaluation of Two-center Moment Integral – An Extension

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In two recent studies [1, 2] we presented a general formulation for the evaluation of two-center moment integrals as well as the rotational properties of s, p, and d atomic orbitals needed in the quantum mechanical calculations of one-electron properties. In the present investigation, we extend the scope of our previous formulation by presenting a series of tabulations useful both to inorganic and organic chemists.

## *(a) One- and Two-center Transition Moment Integrals*

Our most recent investigation of the intensities in the spectrum of $XeF_4$ (l. c. [3]) has revealed certain useful internal relationships existant in one- and two-center transition moments. These are conveniently reported in Tables 1 and 2. It should be noticed that, in the case of the two-center integrals,

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